Technical Notes

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A Computer Survey of Impulsive Ellipse-Ellipse Transfer

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Nomenclature

c_1,c_2	=	constants over the complete multi-impulse trajectory
e	=	orbital eccentricity
f	=	true anomaly, $\theta - \omega$
$F_r,F_\theta,F_\perp$	=	radial, circumferential, and out-of-plane components, respectively, of unit vector parallel to thrust
h	=	angular momentum
H	=	generalized Hamiltonian defined by Eq. (1)
$H_r,H_\theta,H_\perp$	=	Lawden's primer vector; see Eq. (2)
i	=	inclination angle of orbit plane
r	=	radius from force center to point of impulse; see Eq. (4)
v	=	change in vehicle's velocity due to impulses
$v_r,v_\theta,v_\perp$	=	radial, circumferential, and out-of-plane velocity components with respect to axes fixed at start of impulse; see Eq. (7)
x_j	=	state variables; h, e, ω, i, Ω
θ	=	argument of latitude at impulse point
λ_j	=	Lagrange multipliers associated with the state variables x_j
μ	=	product of the central attracting mass and the universal gravitational constant
ω	=	argument of pericenter
Ω	=	longitude of ascending node
$(\ )_a$	=	value of a variable at start of an impulse
$(\ )_0$	=	value of a variable at initial point of multi-impulse trajectory

I. Introduction

THE problem studied is that of minimum impulse (or fuel) transfer of a space vehicle between noncoplanar ellipses. The transfer time, the number of impulses, and the points of departure and arrival on the terminal orbits are all unspecified. The literature for this problem has been surveyed in Refs. 1–3.

The minimizing trajectories can be classified according to whether they require finite or infinite time. The latter maneuvers (called biparabolic) are well understood and use two infinitesimal impulses at $r = \infty$ and two nonvanishing impulses at the pericenters of the terminal orbits. The present Note will be primarily concerned with the number of impulses used by the finite-time maneuvers. When the ellipses are coplanar it is known that this number is not above three.[2,3] The finite-time solutions to the particular noncoplanar configurations that have been studied also employ at most three impulses. The conjecture that there are no finite-time transfers that require more than three impulses may be found in Refs. 3 and 4.

These studies were concerned with maneuvers that are optimal in a global rather than a local sense. In accordance with the theory of the second variation, a member of a family of extremals with common initial point is locally optimal until it contacts an envelope at either a conjugate or a reflection point[5] (see Fig. 1). Darboux[6] has shown that the point at which an extremal ceases to be globally optimal does not as a rule coincide with the envelope contact but rather precedes it. Thus even if the aforementioned conjecture of Refs. 3 and 4 were to be established, there might still be locally optimal finite-time transfer with four impulses. Knowledge of these maneuvers would be particularly valuable whenever the globally optimal trajectory requires infinite time.

The objectives of the present Note are 1) to show that despite the complexity of the problem and the presence of corners on the extremals, it is almost always feasible to test for conjugate and reflection points without using excessive computer time; and 2) to discuss the results of a computer survey consisting of approximately 650 trajectories that were stopped when they contacted an envelope.

II. Analysis

The equations in Mayer format for optimal impulsive orbit transfer are derived from the following generalized Hamiltonian:

$$H = F_r H_r + F_\theta H_\theta + F_\perp H_\perp \tag{1}$$

$$H_r = [h/\mu][\lambda_e \sin f - \lambda_\omega(\cos f/e)] \tag{2a}$$

$$H_\theta = \lambda_h r + [r/h][\lambda_e(2\cos f + e + e\cos^2 f) + \lambda_\omega(2/e + \cos f)\sin f] \tag{2b}$$

$$H_\perp = (r/h)[-\lambda_\omega \sin\theta \cot i + \lambda_i \cos\theta + \lambda_\Omega \sin\theta/\sin i] \tag{2c}$$

$$f = \theta - \omega \tag{3}$$

$$r = h^2/[\mu(1 + e\cos f)] \tag{4}$$

The state variable ($x_1 = h$, $x_2 = e$, $x_3 = \omega$, $x_4 = i$, $x_5 = \Omega$) and Lagrange multiplier derivatives with respect to v are given by†

$$dx_j/dv = \partial H/\partial\lambda_j,\ d\lambda_j/dv = -\partial H/\partial x_j \qquad j = 1, \ldots, 5 \tag{5}$$

In accordance with the principle of Weierstrass-Pontryagin, the control variables ($F_r,F_\theta,F_\perp,\theta$) are chosen so that the Hamiltonian is a maximum. This requires

$$F_r = H_r/H,\ F_\theta = H_\theta/H,\ F_\perp = H_\perp/H \tag{6}$$

After substituting Eq. (6) into Eq. (1) the Hamiltonian must

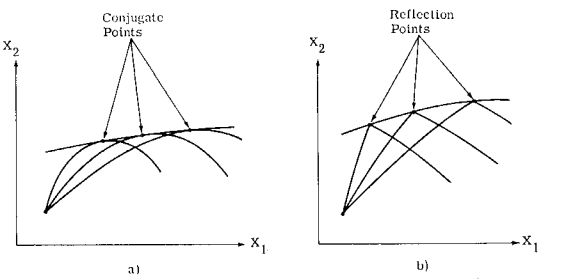

Fig. 1 Extremal families with a) conjugate and b) reflection points.

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† See Ref. 8 for an explicit form of the state variable equations.

still be maximized with respect to the control variable θ. This can be accomplished only by numerical methods.

Given the position, velocity, and thrust direction at the start of an impulse, we wish to find the orbital elements and Lagrange multipliers as the impulse proceeds. First the components of the velocity are found along fixed axes determined by the orbit at the start of the impulse

$$v_r = \mu e_a \sin f_a/h_a + F_{ra}(v - v_a) \tag{7a}$$

$$v_\theta = h_a/r + F_{\theta a}(v - v_a) \tag{7b}$$

$$v_\perp = F_{\perp a}(v - v_a) \tag{7c}$$

Quantities evaluated at the start of the impulse are given the subscript "a." Note that $r = r_a$.

We now employ the standard technique for orbit determination from position and velocity.[7] The angular momentum is easily found from

$$h^2 = r^2(v_\theta^2 + v_\perp^2) \tag{8}$$

Equations (4) and (9) determine e and f

$$hv_r = \mu e \sin f \tag{9}$$

Since by definition the inclination angle i must lie in the first or second quadrant, it is specified by the single equation

$$h \cos i = r(v_\theta \cos i_a - v_\perp \cos\theta_a \sin i_a) \tag{10}$$

The angle Ω is determined by

$$\tan\Omega = \frac{v_\theta \sin i_a \sin\Omega_a + v_\perp(\sin\theta_a \cos\Omega_a + \cos\theta_a \cos i_a \sin\Omega_a)}{v_\theta \sin i_a \cos\Omega_a - v_\perp(\sin\theta_a \sin\Omega_a - \cos\theta_a \cos i_a \cos\Omega_a)} \tag{11}$$

with the quadrant specified by the numerator and denominator. The angle θ can now be found from

$$\sin\theta = \sin i_a \sin\theta_a/\sin i \tag{12}$$

$$\cos\theta = \cos\Omega(\cos\Omega_a \cos\theta_a - \cos i_a \sin\Omega_a \sin\theta_a) + \sin\Omega(\sin\Omega_a \cos\theta_a + \cos i_a \cos\Omega_a \sin\theta_a) \tag{13}$$

and the argument of percenter ω from Eq. (3). Four of the Lagrange multipliers are defined by[8,9]

$$\lambda_h = (-Hv + \lambda_{h_0} h_0)/h \tag{14}$$

$$\lambda_\omega = \lambda_{\Omega_0} \cos i + (c_1 \sin\Omega - c_2 \cos\Omega) \sin i \tag{15}$$

$$\lambda_i = c_2 \sin\Omega + c_1 \cos\Omega \tag{16}$$

$$\lambda_\Omega = \lambda_{\Omega_0} \tag{17}$$

Here c_1 and c_2 are constant over the entire multi-impulse trajectory and can be evaluated by setting i, Ω, λ_ω, and λ_i to their initial values in Eqs. (15) and (16). The final multiplier λ_e could be obtained from the integral H = const, but this equation is highly complex. It could also be obtained from Eq. (2a) since $H_r = HF_r$ does not change during an impulse. However, this equation requires division by $\sin f$ so that it could not be used for the important class of trajectories that employ impulses at apses. Therefore, the following equation was used whenever $\sin f$ was small.

$$\frac{h_a}{r^2}\left(\frac{\partial H_r}{\partial\theta} - H_\theta\right)_a = \frac{-1}{r}\left[\lambda_h h + (e + \cos f)\lambda_e + \frac{\lambda_\omega}{e}\sin f\right] \tag{18}$$

This is the radial component of the time derivative of the 3-vector $(H_r, H_\theta, H_\perp)$ (Lawden's primer vector) and is constant over an impulse.[10]

Analytical expressions for all but two steps in the calculation of an optimal trajectory are now available. Numerical methods must be used only to locate the values of the control variable θ that make H a local maximum and to find the value of v for which H has two equal maximums. After this v (corner point) has been found the quantities subscripted with "a" in Eqs. (7–18) are reset to their new values and the trajectory during the next impulse (subarc) is calculated. This procedure generates only the finite-time transfers since certain artifices must be introduced in order to obtain impulses at $r = \infty$.

Each trajectory was stopped when it ceased to be locally optimal, i.e., at its first point of contact with an envelope. Procedures for locating such points are given in books that discuss the Jacobi condition[6]; the specific technique used in the present study may be found in Ref. 11. This procedure required four linearly independent sets of $\delta x_j(v) (j = 1, \ldots, 5)$. Considerable simplification was achieved by approximating the $\delta x_j(v)$ with $\delta x_j{}^k(v) \simeq x_j{}^k(v) - x_j{}^l(v)$ $(j = 1, \ldots, 5)$ where $x_j{}^l(v)$ represented the nominal trajectory and $x_j{}^k(v)$ was close but non-neighboring. The approximation was validated by means of an internal check described in Ref. 11. Thus the investigation of each nominal trajectory required the computation of five trajectories.

Difficulty arose when the Hamiltonian—considered as a function of θ only—had three nearly equal maximums at the initial point or a corner; so that all of the five trajectories did not use corresponding maximums. Maneuvers that were close to a member of the Contensou-Marchal[4] family of non-coplanar trajectories with chattering control were also troublesome. These singular trajectories usually occur when $\lambda_{\omega_0} = \lambda_{i_0} = \omega_0 = 0$, $\lambda_{\Omega_0} \neq 0$, $i_0 = 90°$. In severe cases this sensitivity combined with computer truncation error meant that even with individualized treatment five close but non-neighboring trajectories could not be calculated. Only a few trajectories of the survey could not be analyzed for these reasons.

III. Computer Survey

Owing to the symmetry of the problem it was unnecessary to use initial values of the state variables other than $h_0 = 1$, $\omega_0 = 0$, $i_0 = 90°$, and $\Omega_0 = 0$. For $e_0 = 0.5$ and 0.9 initial Lagrange multipliers were chosen whose end points were uniformly distributed over half of a unit hypersphere. The symmetry made negative values of λ_{Ω_0} unnecessary. There were 174 (nominal) trajectories in each of these families. Trajectories were also computed for $e_0 = 0.1$, 0.25, and 0.95 bringing the total to approximately 650.

The survey was undertaken with the expectation that locally optimal finite-time maneuvers that employ more than three impulses would be revealed. However, along each of the trajectories there was an envelope contact at or before the end of the third impulse. We conclude that exceptions either do not exist or are relatively rare.

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Supersonic Molecular Beams with Cycling-Pressure Sources

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Introduction

SUPERSONIC molecular beams that sample from steady sources have been studied analytically and experimentally for two decades. On the other hand, molecular-beam sampling from an unsteady source has not been studied before.¶ Possible motivations for such studies include a) general investigations of chemical reaction rates and b) specific investigations of chemical processes in combustion chambers of reciprocating internal-combustion engines, useful in efforts to reduce air pollution due to emissions from automobile engines. Both motivations contributed to the initiation of the studies described here.

The investigations of steady supersonic molecular beams indicate that, for a single-species beam operating under ideal conditions (no skimmer interactions and no attenuations due to background gases), the beam density is proportional directly to the source density. It is found[2,3] that skimmer interference depends on stagnation-chamber conditions, source-chamber conditions, and nozzle-skimmer distance. For a cycling-pressure source, both stagnation-chamber conditions and source-chamber conditions change during the cycle with the result that a given nozzle-skimmer distance is not optimum for all parts of the cycle. A compromise nozzle-skimmer distance might be required.

The main purpose of the present study was to investigate those problems which are unique to a cycling-pressure source. Hence, a single inert species was used. Studies made using inert mixtures and reactive mixtures will be described in subsequent publications.[4–6]

Analysis of a Simplified Model

The beam source used in this study is a small reciprocating engine driven by a synchronous electric motor. During the engine cycle, some of the working gas is expanded through a small orifice located at the center of the cylinder head into a vacuum chamber called the source chamber. A supersonic molecular beam is formed by use of a conventional skimmer and a collimation orifice located downstream of the nozzle (see Fig. 1, Ref. 4).

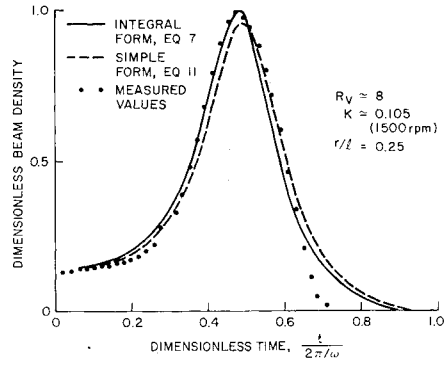

Fig. 1 Comparison between predicted and measured beam densities: The peak of the measured signal is matched to the peak calculated from Eq. (7). The time abscissa is normalized using the engine period.

A. *Temporal state of the source gas*

The temporal state of the gas inside the engine cylinder was considered for a simplified model in which 1) the velocity of the piston is negligible in comparison with the sound speed of the gas inside the cylinder (hence pressure gradients are not considered); 2) heat transfer between the gas and the cylinder wall is negligible; 3) the effective orifice diameter d^* remains constant throughout the entire cycle; 4) the gas is inert chemically; 5) the gas is perfect thermally and calorically; and 6) the square of the crankshaft radius r divided by the connecting-rod length l is small in comparison with unity. For this model, one may write

$$pV = MRT \tag{1}$$

$$p/T^{\gamma/\gamma-1} = \text{const} \tag{2}$$

$$V \approx V_i - \tfrac{1}{2}V_i(1 - 1/R_v)(1 - \cos\theta - \tfrac{1}{2}(r/l)\sin^2\theta) \tag{3}$$

$$\frac{dT}{d\theta} \approx \frac{(\gamma - 1)T}{2} \frac{(1-1/R_v)[\sin\theta - \tfrac{1}{2}(r/l)\sin 2\theta] - K(T/T_i)^{1/2}}{1 - \tfrac{1}{2}(1 - 1/R_v)[1 - \cos\theta - \tfrac{1}{2}(r/l)\sin^2\theta]} \tag{4}$$

where p, V, M, R, T, and γ are respectively the pressure, volume, mass, gas constant, temperature, and specific-heat ratio of the gas inside the engine cylinder, θ is the angular position of the crankshaft measured from bottom dead center, V_i is the volume at $\theta = 0$, R_v is the volume compression ratio, ω is the angular speed of the crankshaft, and K is a dimensionless effusion parameter defined by

$$K \equiv (\pi/2)\{\gamma[2/(\gamma + 1)]^{(\gamma+1)/(\gamma-1)}RT_i\}^{1/2}(d^{*2}/\omega V_i) \tag{5}$$

Equation (4) emphasizes that the temperature variation is due to a) the piston motion and b) the gas escaping through the nozzle.

Using Eq. (4), one can determine $T(\theta)$ by numerical integration. Using Eqs. (1–3), one can determine V, p, and M then also as functions of θ.

B. *Predicted beam density at the detector*

For a beam with a cycling-pressure source, differences in molecule flight times from the sudden-freezing surface to the detector imply (at least in principle) that molecules detected at the detector at a given time came from different parts of the source cycle. Let bottom dead center of the crankshaft correspond to time zero. Then, at time λ, the number of molecules per unit volume in the speed-ratio range from s to

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¶ Note that the pulsed beam generated by Bier and Hagena[1] used a steady-state source. The beam was pulsed by opening and closing a valve at the nozzle rather than by varying the source condition.